Indian Statistical Institute

B. Math. I Year

Mid-Semestral Examination 2008-2009

Probability Theory I

Date: 23-09-2008 Total Marks:100 Instructor: B. Rajeev

1. Let $S = \{1, 2, \ldots, n\}$ and suppose that A and B are chosen from the power set of S ,independently, and equally likely to be any of the 2^n subsets (including the null set and S itself) of S.
   (a) Show that
$$P\{A \subset B\} = \left(\frac{3}{4}\right)^n.$$
   Hint: Condition on the number of elements in B (10)

2. Suppose that n independent trials, each of which results in any of the outcomes 0, 1, or 2 with respective probabilities, p_0, p_1, and p_2, $\sum_{i=0}^{2} p_i = 1$, are performed. Find an expression for the probability that outcomes 1 and 2 both occur at least once. (15)

3. Independent Bernoulli trials with probability of success p are performed. Let P_n denote the probability that n Bernoulli trials result in an even number of successes (0 being considered an even number). Prove that
$$P_n = \frac{1 + (1 - 2p)^n}{2}.$$
   (20)

4. Let X be a binomial random variable with parameters n and p. Show that
$$E\left[\frac{1}{X+1}\right] = \frac{1 - (1-p)^{n+1}}{(n+1)p}.$$
   (10)

5. If X is a Poisson random variable with parameter λ, show that

$$E[X^n] = \lambda E[(X+1)^{n-1}]$$

Use this result to compute $E[X^3]$. (5+5)

6. From a set of n elements a nonempty subset is chosen at random in the sense that all of the nonempty subsets are equally likely to be selected. Let X denote the number of elements in the chosen subset. Show that

$$\begin{aligned} E[X] &= \frac{n}{2-(\frac{1}{2})^{n-1}} \\ \mathrm{Var}(X) &= \frac{n \cdot 2^{2n-2} - n(n+1)2^{n-2}}{(2^n-1)^2} \end{aligned}$$

(10 + 10)

7. An urn initially contains one red and one blue ball. At each stage a ball is randomly chosen and then replaced along with another of the same color. For X denote the selection number of the first chosen ball that is blue. For instance, if the first selection is red and the second blue, then X is equal to 2.
(a) Find $P\{X > i\}$, $i \geq 1$.
(b) Show that with probability 1, a blue ball is eventually chosen. (That is, show that $P\{X < \infty\} = 1$.)
(c) Find $E[X]$. (5 + 6 + 4)